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Stability Analysis of Polynomial-Fuzzy-Model-Based Control Systems with Mismatched Premise Membership Functions

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***Abstract*—This paper investigates the stability of polynomial-fuzzy-model-based (PFMB) control system, which is formed by a polynomial fuzzy model and a polynomial fuzzy controller connected in a closed loop. To enhance the design flexibility, the number of rules and the shape of premise membership functions of the polynomial fuzzy controller are considered to be chosen freely and different from those of the polynomial fuzzy model, however, which make the stability analysis more difficulty and potentially lead to conservative stability analysis result. A sum-of-squares (SOS)-based stability analysis approach using the Lyapunov stability theory is proposed to investigate the stability of the PFMB control systems and synthesize the polynomial fuzzy controller. To facilitate the stability analysis and relax the stability analysis result, the property of the membership functions, and the boundary information of the membership grades and premise variables are taken into account in the stability analysis and incorporated into the SOS-based stability conditions. A simulation example is given to illustrate the effectiveness of the proposed approach.**

***Index Terms*—Fuzzy-Model-Based Control, Mismatched Premise Membership Functions, Nonlinear Systems, Polynomial Fuzzy Systems, Stability Analysis, Sum-of-Squares.**

I. INTRODUCTION

FUZZY-model-based (FMB) control has been a promise research topic for the past two decades [1]. The T-S fuzzy model [2], [3] nailed down the fundamental platform for development of FMB control approach and theoretical basis for analysis of FMB control systems. A class of nonlinear system can be represented by the T-S fuzzy model in a favourable form to facilitate the stability analysis and control synthesis. Basic and improved linear-matrix-inequality (LMI)-based stability conditions for the FMB control systems with fuzzy state-feedback controllers (referred to as fuzzy controller hereafter) under the parallel distributed compensation (PDC) design concept were achieved in [4]–[12]. A feasible solution to the LMI-based conditions can be found numerically using convex programming techniques [13]. With the consideration of the information of membership functions, various relaxed stability analysis results were obtained [14]–[23].

The PDC design concept proposes that the fuzzy controller shares the same premise membership functions and the same number of rules as those of the T-S fuzzy model. Although the PDC design concept makes the stability analysis easier for the FMB control systems due to the matched premise membership functions resulting in more relaxed stability conditions, it limits the design flexibility of the fuzzy controller and complicates its structure in some cases leading to a higher implementation cost, particularly, when the membership functions of the T-S fuzzy model are complicated and/or the number of rules is large. It motivates the non-PDC design concept for the FMB control systems. In this paper, the non-PDC design is referred to the case that the fuzzy controller does not share the same premise membership functions and/or the same number of rules as those of the T-S fuzzy model. Not much work has been found in the literature for the FMB control system under the non-PDC design concept since the mismatched premise membership functions and/or number of rules will make the stability analysis more difficult and lead to potentially conservative stability analysis result. The information of the membership functions plays an important role in the non-PDC-based stability analysis and to relax the stability analysis result [14], [15], [17], [18], [22], [23].

Recently, the T-S fuzzy model has been extended to polynomial fuzzy model [24]–[26] representing a wider class of nonlinear plants with the consideration of polynomials in the local models. Similarly, a polynomial fuzzy controller, which allows polynomials in the feedback gains, was proposed for the control process. The capabilities of modelling and feedback compensation are enhanced but the existence of polynomial variables hinders the stability analysis when the LMI-based analysis approach is applied. Instead, the sum-of-squares (SOS) approach [27] was then employed to investigate the stability of polynomial-fuzzy-model-based (PFMB) control systems [24], [25]. Based on the Lyapunov stability theory, basic PDC SOS-based stability conditions [24], [25] were obtained. A feasible solution to the SOS-based stability conditions can be found numerically using the third-party Matlab toolbox SOSTOOLS [28]. Relaxed stability analysis results can be found in [26], [29]–[31]. The work in [26] was based on PDC design concept of which the technique of variable transformation was employed for the stability analysis. The work in [29]–[31] considered the non-PDC design concept of which the technique of membership function approximation was employed for stability analysis.

In this paper, we investigate the stability of PFMB control systems under the non-PDC design concept based on the SOS-based approach using the Lyapunov stability theory. In [29],

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[31], this class of PFMB control systems was investigated through approximated membership functions, which approximate the original ones. The effectiveness of the stability analysis result depends on the approximation accuracy. Higher approximation accuracy in general will result in a larger number of SOS-based stability conditions, which implies a higher computational demand for solution search process. In [30], the stability analysis was carried out for the PFMB control system of which the polynomial fuzzy model and polynomial fuzzy controller share the same number of rules but with different premise membership functions. This paper will base on the work in [30] and further relax the constraints such that both premise membership functions and the number of rules are allowed to be freely chosen. A promising stability analysis approach is proposed to investigate the stability of the PFMB control systems by considering three pieces of information, namely, the property of membership functions, and the boundary information of membership grades and premise variables.

The rest of this paper is organized as follows. In Section II, notations used in this paper are introduced. The details of the polynomial fuzzy model and polynomial fuzzy controller are presented. In Section III, SOS-based stability conditions are obtained for the PFMB control system under the non-PDC design concept using the Lyapunov stability theory. In Section IV, a simulation example is given to illustrate the merits of the proposed control scheme. In section V, a conclusion is drawn.

II. Notations and Preliminaries

A. Notation

Throughout the paper, the following notations are adopted [27]. A monomial in $\mathbf{x}(t) = [x_1(t), \cdots, x_n(t)]$ is a function of the form $x_1^{d_1}(t) \cdots x_n^{d_n}(t)$ where d_i, $i = 1, \cdots, n$, are nonnegative integers. The degree of a monomial is defined as $d = \sum_{i=1}^{n} d_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is defined as a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is a sum of squares if it can be written as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$ where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a non-zero positive integer. Hence, it can be seen that $\mathbf{p}(\mathbf{x}(t)) \geq 0$ if it is an SOS. The expressions of $\mathbf{M} > 0$, $\mathbf{M} \geq 0$, $\mathbf{M} < 0$ and $\mathbf{M} \leq 0$ denote the positive, semi-positive, negative and semi-negative definite matrices $\mathbf{M}$, respectively.

B. Polynomial Fuzzy Model

Let p be the number of fuzzy rules describing the behavior of a nonlinear plant [24], [25]. The i-th rule is of the following format:

$$\begin{aligned}&\text{Rule } i\text{: IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \\ &\qquad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \quad (1)\end{aligned}$$

where M_α^i is the fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, \cdots, \Psi$; $i = 1, \cdots, p$; Ψ is a positive integer; $\mathbf{x}(t) \in \Re^n$ is the system state vector; $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n \times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \Re^{n \times m}$ are the known polynomial system and input matrices, respectively; $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}(t)$; $\mathbf{u}(t) \in \Re^m$ is the input vector. It is assumed that $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$ if and only if $\mathbf{x}(t) = \mathbf{0}$. The system dynamics is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\big), \quad (2)$$

where $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1$, $w_i(\mathbf{x}(t)) \geq 0$, $w_i(\mathbf{x}(t)) = \dfrac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^{p}\prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(t)))}$ for all i, $w_i(\mathbf{x}(t))$ is the normalized grade of membership; $\mu_{M_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, \cdots, \Psi$, is the grade of membership corresponding to the fuzzy term M_α^i.

C. Polynomial Fuzzy Controller

A polynomial fuzzy controller is proposed to stabilize the nonlinear plant represented by the polynomial fuzzy model (2). The polynomial fuzzy controller is described by the following c rules.

$$\begin{aligned}&\text{Rule } j\text{: IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND } \cdots \text{ AND } g_\Omega(\mathbf{x}(t)) \text{ is } N_\Omega^j \\ &\qquad \text{THEN } \mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \quad (3)\end{aligned}$$

where N_β^j is the fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, \cdots, \Omega$; $j = 1, \cdots, c$; Ω is a positive integer; $\mathbf{G}_j(\mathbf{x}(t)) \in \Re^{m \times N}$, $j = 1, \cdots, c$, is the polynomial feedback gain to be determined. The polynomial fuzzy controller is defined as follows.

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \quad (4)$$

where $\sum_{j=1}^{c} m_i(\mathbf{x}(t)) = 1$, $m_j(\mathbf{x}(t)) \geq 0$, $m_j(\mathbf{x}(t)) = \dfrac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(t)))}{\sum_{k=1}^{c}\prod_{l=1}^{\Omega} \mu_{N_l^k}(g_l(\mathbf{x}(t)))}$ for all j, $m_j(\mathbf{x}(t))$ is the normalized grade of membership; $\mu_{N_\alpha^j}(g_\alpha(\mathbf{x}(t)))$, $\beta = 1, \cdots, \Omega$, is the grade of membership corresponding to the fuzzy term N_β^j.

III. Stability Analysis

A PFMB control system is formed by the polynomial fuzzy model (2) and polynomial fuzzy controller (4) connected in a closed loop. In the following analysis, for brevity, the time t associated with the variables is dropped for the situation without ambiguity, e.g., $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are denoted as $\mathbf{x}$ and $\mathbf{u}$, respectively. Furthermore, $\hat{\mathbf{x}}(\mathbf{x}(t))$, $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ are denoted as $\hat{\mathbf{x}}$, w_i and m_j, respectively.

From (2) and (4), a PFMB control system is obtained as follows.

$$\begin{aligned}\dot{\mathbf{x}} &= \sum_{i=1}^{p} w_i\Big(\mathbf{A}_i(\mathbf{x})\hat{\mathbf{x}} + \mathbf{B}_i(\mathbf{x})\sum_{j=1}^{c} m_j\mathbf{G}_j(\mathbf{x})\hat{\mathbf{x}}\Big) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j\big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x})\big)\hat{\mathbf{x}} \quad (5)\end{aligned}$$

The control objective is to determine the polynomial feedback gains $\mathbf{G}_j(\mathbf{x})$ such that the PFMB system (5) is asymptotically stable i.e., $\mathbf{x}(t) \to \mathbf{0}$ as time $t \to \infty$.

To proceed with the stability analysis, we denote $\mathbf{x} = [x_1, \cdots, x_n]^T$ and $\hat{\mathbf{x}} = [\hat{x}_1, \cdots, \hat{x}_N]^T$. From (5), we have

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} \\ &= \mathbf{T}(\mathbf{x})\dot{\mathbf{x}} \\ &= \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \big(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x})\big)\hat{\mathbf{x}}, \qquad (6)\end{aligned}$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$, $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$, $\mathbf{T}(\mathbf{x}) \in \Re^{N \times n}$ with its (i,j)th entry given as $\frac{\partial \hat{x}_i(\mathbf{x})}{\partial x_j}$.

Because of the assumption $\hat{\mathbf{x}} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$, the stability of the PFMB control system (6) implies that of (5). We consider the following polynomial Lyapunov function candidate to investigate the system stability of (6),

$$V(\mathbf{x}) = \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}, \qquad (7)$$

where $0 < \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N \times N}$ is a polynomial matrix.

From (6) and (7), we have,

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \dot{\hat{\mathbf{x}}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{x}} + \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1}\dot{\hat{\mathbf{x}}} + \hat{\mathbf{x}}^T \dot{\mathbf{X}}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{x}} \\ &= \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \hat{\mathbf{x}}^T \Big(\big(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x})\big)^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ &\quad + \mathbf{X}(\tilde{\mathbf{x}})^{-1} \big(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x})\big)\Big)\hat{\mathbf{x}} \\ &\quad + \hat{\mathbf{x}}^T \dot{\mathbf{X}}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{x}}. \qquad (8)\end{aligned}$$

Remark 1: To facilitate the stability analysis [24], [27], $\mathbf{K} = \{k_1, \cdots, k_q\}$ is defined as the set of row number that the entries of the entire row of $\mathbf{B}_i(\mathbf{x})$ are all zero for all i.

Lemma 1 ([24], [27]): For any invertible polynomial matrix $\mathbf{X}(\mathbf{z})$ where $\mathbf{z} = (z_1, \cdots, z_n)$, the following equation is true.

$$\frac{\partial \mathbf{X}(\mathbf{z})^{-1}}{\partial z_j} = -\mathbf{X}(\mathbf{z})^{-1} \frac{\partial \mathbf{X}(\mathbf{z})}{\partial z_j} \mathbf{X}(\mathbf{z})^{-1} \; \forall \, j$$

Define $\mathbf{z} = \mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{x}}$ and $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$ where $\mathbf{N}_j(\mathbf{x}) \in \Re^{m \times N}$, $j = 1, \cdots, c$, is an arbitrary polynomial matrix. From (8), with Remark 1 and Lemma 1, we have

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \mathbf{z}^T \mathbf{Q}_{ij}(\mathbf{x})\mathbf{z}, \qquad (9)$$

where $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}$ for $i = 1, \cdots, p$; $j = 1, \cdots, c$; $\mathbf{A}_i^k(\mathbf{x})$ denotes the k^{th} row of $\mathbf{A}_i(\mathbf{x})$.

Based on the Lyapunov stability theory, $V(\mathbf{x}) > 0$ and $\dot{V}(\mathbf{x}) < 0$ (excluding $\mathbf{x} = \mathbf{0}$) imply the asymptotic stability of (6) which can be achieved by $\mathbf{Q}_{ij}(\mathbf{x}) < 0$ for all i and j. The stability analysis result is summarized in the following theorem.

Theorem 1: The PFMB control system (5) formed by a nonlinear plant represented by the polynomial fuzzy model in the form of (2) and a polynomial fuzzy controller (4) connected in a closed loop, is asymptotically stable if there exist polynomial matrices $\mathbf{N}_j(\mathbf{x}) \in \Re^{m \times N}$, $j = 1, \cdots, c$ and $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N \times N}$, such that the following SOS-based conditions are satisfied.

$$\nu^T \big(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I}\big)\nu \text{ is SOS;}$$
$$-\nu^T \big(\mathbf{Q}_{ij}(\mathbf{x}) + \varepsilon_2(\mathbf{x})\mathbf{I}\big)\nu \text{ is SOS, } \forall \, i, j;$$

where $\nu \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}$; $\varepsilon_1(\tilde{\mathbf{x}}) > 0$ and $\varepsilon_2(\mathbf{x}) > 0$ are predefined scalar polynomials; and the polynomial feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$.

Remark 2: Theorem 1 provides the basic non-PDC SOS-based stability conditions to synthesize the polynomial fuzzy controller with the consideration of the stability of the PFMB control system. However, it is very conservative because the information of the membership functions, w_i and m_j, are not considered. In [24], the SOS-based stability conditions can only be applied to the PFMB control systems under the PDC design concept.

In the following, we propose a stability analysis approach for relaxing the basic non-PDC SOS-based stability conditions in Theorem 1 by considering the property of membership functions, and the boundary information of members grades and premise variables in the stability analysis through some slack matrices.

Define a vector $\xi = \begin{bmatrix} \mathbf{w} & \mathbf{m} & \mathbf{x} \end{bmatrix}$ where $\mathbf{w} = \begin{bmatrix} w_1 & \cdots & w_p \end{bmatrix}$ and $\mathbf{m} = \begin{bmatrix} m_1 & \cdots & m_c \end{bmatrix}$. We consider the scalar polynomial functions $\gamma_{1h_1}(\xi) = 0$, $\gamma_{2h_2}(\xi) \geq 0$ and $\gamma_{3h_3}(\xi) \geq 0$, $h_1 = 1, \cdots, H_1$, $h_2 = 1, \cdots, H_2$, $h_3 = 1, \cdots, H_3$, incorporating the property of membership functions, and the boundary information of members grades and premise variables, respectively.

From (9), we have

$$\begin{aligned}\dot{V}(\mathbf{x}) &\leq \sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \mathbf{z}^T \mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} \\ &\quad + \sum_{k=1}^{3} \sum_{h_k=1}^{H_k} \gamma_{kh_k}(\xi)\mathbf{z}^T \mathbf{R}_{kh_k}(\xi)\mathbf{z} \\ &= \mathbf{z}^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \mathbf{Q}_{ij}(\mathbf{x}) \\ &\quad + \sum_{k=1}^{3} \sum_{h_k=1}^{H_k} \gamma_{kh_k}(\xi)\mathbf{R}_{kh_k}(\xi)\Big)\mathbf{z}, \qquad (10)\end{aligned}$$

where $\mathbf{R}_{1h_1}(\xi) = \mathbf{R}_{1h_1}(\xi)^T \in \Re^{N \times N}$ is an arbitrary slack polynomial matrix, $0 \leq \mathbf{R}_{2h_2}(\xi) = \mathbf{R}_{2h_2}(\xi)^T \in \Re^{N \times N}$ and $0 \leq \mathbf{R}_{3h_3}(\xi) = \mathbf{R}_{3h_3}(\xi)^T \in \Re^{N \times N}$ are slack polynomial matrices.

Based on the Lyapunov stability theory, $V(\mathbf{x}) > 0$ and $\dot{V}(\mathbf{x}) < 0$ (excluding $\mathbf{x} = \mathbf{0}$) imply the asymptotic stability of (6) which can be achieved by $\sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \mathbf{Q}_{ij}(\mathbf{x}) + \sum_{k=1}^{3} \sum_{h_k=1}^{H_k} \gamma_{kh_k}(\xi)\mathbf{R}_{kh_k}(\xi) < 0$. The stability analysis result is summarized in the following theorem.

Theorem 2: The PFMB control system (5) formed by a nonlinear plant represented by the polynomial fuzzy model in the form of (2) and a polynomial fuzzy controller (4) connected in a closed loop, is asymptotically stable if there exist polynomial matrices $\mathbf{N}_j(\mathbf{x}) \in \Re^{m\times N}$, $j = 1, \cdots, c$, $\mathbf{R}_{1h_1}(\xi) = \mathbf{R}_{1h_1}(\xi)^T \in \Re^{N\times N}$, $h_1 = 1, \cdots, H_1$, $\mathbf{R}_{2h_2}(\xi) = \mathbf{R}_{2h_2}(\xi)^T \in \Re^{N\times N}$, $h_2 = 1, \cdots, H_2$, $\mathbf{R}_{3h_3}(\xi) = \mathbf{R}_{3h_3}(\xi)^T \in \Re^{N\times N}$, $h_3 = 1, \cdots, H_3$, and $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$, such that the following SOS-based conditions are satisfied.

$$\nu^T\big(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I}\big)\nu \text{ is SOS;}$$
$$\nu^T\big(\mathbf{R}_{2h_2}(\xi) - \varepsilon_{2h_2}(\xi)\mathbf{I}\big)\nu \text{ is SOS}, \forall\, h_2;$$
$$\nu^T\big(\mathbf{R}_{3h_3}(\xi) - \varepsilon_{3h_3}(\xi)\mathbf{I}\big)\nu \text{ is SOS}, \forall\, h_3;$$

$$-\nu^T\big(\sum_{i=1}^{p}\sum_{j=1}^{c} w_i m_j \mathbf{Q}_{ij}(\mathbf{x}) + \sum_{k=1}^{3}\sum_{h_k=1}^{H_k}\gamma_{kh_k}(\xi)\mathbf{R}_{kh_k}(\xi) + \varepsilon_4(\xi)\mathbf{I}\big)\nu \text{ is SOS, } \forall\, i, j;$$

where $\nu \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}$; $\varepsilon_1(\tilde{\mathbf{x}}) > 0$, $\varepsilon_{2h_2}(\xi)$, $\varepsilon_{3h_3}(\xi)$ and $\varepsilon_4(\xi) > 0$ are predefined scalar polynomials; and the polynomial feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$.

Remark 3: It should be noted that $\mathbf{w}$, $\mathbf{m}$ and $\mathbf{x}$ are considered as symbolic variables when the third-party Matlab toolbox SOSTOOLS [28] is employed to search for a feasible solution to the SOS-based stability conditions in Theorem 2.

Remark 4: To reduce the computational demand for the solution search process, the number of symbolic variables can be reduced by using the property of the membership functions of (2) and (4), i.e., $w_p = 1-\sum_{i=1}^{p-1} w_i$ and $m_c = 1-\sum_{j=1}^{c-1} m_j$.

In the following, with the consideration of Remark 3, further details regarding the property of membership functions, and the boundary information of members grades and premise variables are discussed.

A. Property of Membership Functions

The property of the membership functions imposes a constraint on the sum of membership grades. $\gamma_{1h_1}(\xi) = 0$ is a scalar polynomial, which is constructed based on the property of the membership functions. For example, we can choose the following scalar polynomials as $\gamma_{1h_1}(\xi)$.

$$\gamma_{1h_1}(\xi) = \sum_{i=1}^{p} w_i - 1 = 0 \tag{11}$$

$$\gamma_{1h_1}(\xi) = \sum_{j=1}^{c} m_j - 1 = 0 \tag{12}$$

$$\gamma_{1h_1}(\xi) = \sum_{i=1}^{p} w_i - \sum_{j=1}^{c} m_j = 0 \tag{13}$$

$$\gamma_{1h_1}(\xi) = \sum_{i=1}^{p}\sum_{=1}^{c} w_i m_j - 1 = 0 \tag{14}$$

A general expression is given below:

$$\gamma_{1h_1}(\xi) = \big(\sum_{i=1}^{p} w_i\big)^{d_1}\big(\sum_{j=1}^{c} m_j\big)^{d_2} - \big(\sum_{k=1}^{p} w_k\big)^{d_3}\big(\sum_{l=1}^{c} m_l\big)^{d_4} = 0, \tag{15}$$

where $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$ and $d_4 \geq 0$ are integers to be determined. By choosing the proper values of d_1 to d_4, (15) can be reduced to (11) to (14). It should be noted that when the technique in Remark 4 is employed, the scalar polynomial $\gamma_{1h_1}(\xi)$ will vanish from the SOS-based stability conditions in Theorem 2 as the symbolic variables w_i and m_j will cancel out each other.

B. Boundary Information of Membership Grades

As different membership functions are employed for different polynomial fuzzy models and polynomial fuzzy controllers, the boundary information of membership grades will give an idea on the shape of membership functions. By incorporating this information into the stability analysis, the SOS-based stability conditions are more dedicated to the PFMB control system under consideration resulting in a more relaxed stability analysis result.

$\gamma_{2h_2}(\xi) \geq 0$ is a scalar polynomial, which carries the boundary information of membership grades to the stability analysis. For example, we can choose the following scalar polynomials as $\gamma_{2h_2}(\xi)$.

$$\gamma_{2h_2}(\xi) = w_i - \underline{w}_i \geq 0 \tag{16}$$

$$\gamma_{2h_2}(\xi) = \overline{w}_i - w_i \geq 0 \tag{17}$$

$$\gamma_{2h_2}(\xi) = m_j - \underline{m}_j \geq 0 \tag{18}$$

$$\gamma_{2h_2}(\xi) = \overline{m}_j - m_j \geq 0 \tag{19}$$

$$\gamma_{2h_2}(\xi) = w_i m_j - \underline{\mu}_{ij} \geq 0 \tag{20}$$

$$\gamma_{2h_2}(\xi) = \overline{\mu}_{ij} - w_i m_j \geq 0 \tag{21}$$

where $\underline{w}_i$ and $\overline{w}_i$ denote the lower and upper bounds of w_i, respectively; $\underline{m}_j$ and $\overline{m}_j$ denote the lower and upper bounds of m_j, respectively; $\underline{\mu}_{ij}$ and $\overline{\mu}_{ij}$ denote the lower and upper bounds of $w_i m_j$, respectively. A general form of $\gamma_{h_2}(\xi)$ is given below:

$$\gamma_{2h_2}(\xi) = (\eta_{2h_2}(\xi) - \underline{\eta}_{2h_2})^{d_1}(\overline{\eta}_{2h_2} - \eta_{2h_2}(\xi))^{d_2} \geq 0, \tag{22}$$

where $\eta_{2h_2}(\xi)$ is a scalar polynomial, $\underline{\eta}_{2h_2}$ and $\overline{\eta}_{2h_2}$ denote the lower and upper bounds of $\eta_{2h_2}(\xi)$, respectively, $d_1 \geq 0$ and $d_2 \geq 0$ are integers to be determined.

C. Boundary Information of Premise Variables

For some membership functions, the membership grades become zero for some domains of premise variables, which provide important information in the stability analysis. $\gamma_{h_3}(\xi) \geq 0$ is a scalar polynomial, which carries the boundary information of premise variables to the stability analysis. A general form of $\gamma_{h_3}(\xi)$ is given below:

$$\gamma_{3h_3}(\xi) = \varphi_{3h_3}(\mathbf{x})\eta_{3h_3}(\xi) \geq 0, \tag{23}$$

where $\eta_{3h_3}(\xi) \geq 0$ is a scalar polynomial and $\varphi_{3h_3}(\mathbf{x})$ is a scalar polynomial in $\mathbf{x}$, which satisfies $\begin{cases} \varphi_{3h_3}(\mathbf{x}) \leq 0 \text{ for } \eta_{3h_3}(\xi) = 0 \\ \varphi_{3h_3}(\mathbf{x}) \geq 0 \text{ for } \eta_{3h_3}(\xi) > 0 \end{cases}$. For example, considering the membership function $m_1(x_1)$ with its membership grade as zero for $x_1 \geq 5$, we can choose $\varphi_{3h_3}(\mathbf{x}) = 5 - x_1$ and $\eta_{3h_3}(\xi) = w_1 m_1$ resulting in $\gamma_{3h_3}(\xi) = (5 - x_1)w_1 m_1$. For $x_1 \leq 5$, as $\varphi_{3h_3}(\mathbf{x}) = 5 - x_1 \geq 0$ and $\eta_{3h_3}(\xi) = w_1 m_1 \geq 0$, it leads to $\gamma_{3h_3}(\xi) = (5 - x_1)w_1 m_1 \geq 0$. For $x_1 \geq 5$, as $5 - x_1 \leq 0$ and $\eta_{3h_3}(\xi) = w_1 m_1 = 0$, it leads to $\gamma_{3h_3}(\xi) = (5 - x_1)w_1 m_1 = 0$. It should be noted from Remark 3 that w_i, m_j and $\mathbf{x}$ are symbolic variables. Although $\gamma_{3h_3}(\xi) = 0$, the boundary information of premise variables can still be brought to the stability analysis.

IV. Simulation Example

In this example, a polynomial fuzzy model with 3 rules in the form of (2) is considered to represent the nonlinear plant with the following parameters:

$$\hat{\mathbf{x}} = \mathbf{x} = [x_1 \quad x_2]^T,$$

$$\mathbf{A}_1(x_1) = \begin{bmatrix} 1.59 + 2.45x_1 & -7.29 - 0.89x_1 \\ 0.01 & -0.1 - 0.27x_1^2 \end{bmatrix},$$

$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.02 - 7.26x_1 - 0.05x_1^2 & -4.64x_1 \\ 0.35 - 0.28x_1 & -0.21 - 1.65x_1^2 \end{bmatrix},$$

$$\mathbf{A}_3(x_1) = \begin{bmatrix} -a + 0.37x_1 - 2.7x_1^2 & -4.33 - 2.73x_1^2 \\ 1.77x_1 & 0.05 - x_1^2 \end{bmatrix},$$

$$\mathbf{B}_1(x_1) = \begin{bmatrix} 1 + 0.37x_1 + 1.28x_1^2 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_2(x_1) = \begin{bmatrix} 8 + 0.23x_1^2 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_3(x_1) = \begin{bmatrix} -b + 6 + 0.72x_1 + 1.55x_1^2 \\ -1 \end{bmatrix},$$

where a and b are constant scalars.

The membership functions of the polynomial fuzzy model are chosen as $w_1(x_1) = \mu_{M_1^1}(x_1) = 1 - \frac{1}{1+e^{-(x_1+3)}}$, $w_2(x_1) = \mu_{M_1^2}(x_1) = 1 - w_1(x_1) - w_3(x_1)$, $w_3(x_1) = \mu_{M_1^3}(x_1) = \frac{1}{1+e^{-(x_1-3)}}$, which are shown in Fig. 1.

A fuzzy controller with 2 rules in the form of (4) is employed to close the feedback loop of the polynomial fuzzy model to form a PFMB control system in the form of (5). The membership functions are chosen as $m_1(x_1) = \mu_{N_1^1}(x_1) = \begin{cases} 1 & \text{for } x_1 < -5 \\ \frac{-x_1+5}{10} & \text{for } -5 \leq x_1 \leq 5 \\ 0 & \text{for } x_1 > 5 \end{cases}$ and $m_2(x_1) = \mu_{N_1^2}(x_1) = 1 - m_1(x_1)$, which are shown in Fig. 1.

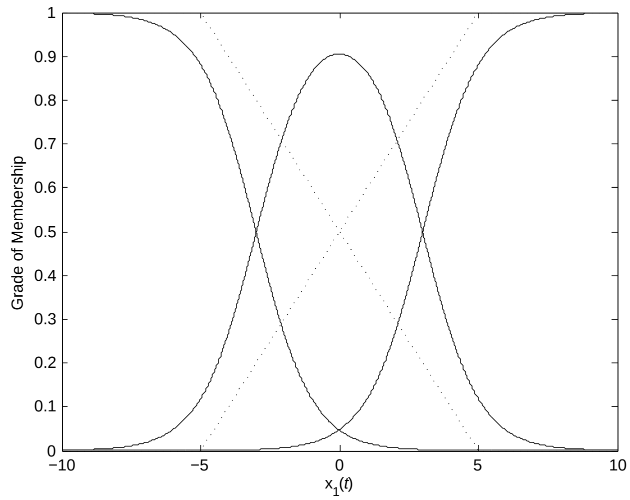

Fig. 1. Membership functions of the polynomial fuzzy model (solid lines) and polynomial fuzzy controller (dotted lines).

In this example, in order to reduce the computational demand on searching for a feasible solution, the technique in Remark 4 is employed, i.e., $w_3(x_1) = 1 - w_1(x_1) - w_2(x_1)$ and $m_2(x_1) = 1 - m_1(x_1)$, which reduce two symbolic variables in the SOS-based stability conditions in Theorem 2. Consequently, the property of the membership functions cannot be used.

The boundary information of membership grades is considered. Denote $\overline{w}_i$ as the upper bound of $w_i(x_1)$; $\overline{m}_j$ as the upper bound of $m_j(x_1)$; $\mu_{ij}(x_1) \equiv w_i(x_1)m_j(x_1)$ and its upper bound as $\overline{\mu}_{ij}$, $i = 1, 2, 3$ and $j = 1, 2$. We consider the following inequalities to bring the boundary information of membership grades to the stability analysis.

$$\begin{aligned} &\gamma_{2h_2}(\xi) = w_i(x_1)(\overline{w}_j - w_j(x_1)) \geq 0, \\ &h_2 = 3(i-1) + j = 1, \cdots, 9; i, j = 1, 2, 3. \end{aligned} \tag{24}$$

$$\begin{aligned} &\gamma_{2h_2}(\xi) = m_i(x_1)(\overline{m}_j - m_j(x_1)) \geq 0, \\ &h_2 = 9 + 2(i-1) + j = 10, \cdots, 13; i, j = 1, 2. \end{aligned} \tag{25}$$

$$\begin{aligned} &\gamma_{2h_2}(\xi) = \overline{\mu}_{ij} - \mu_{ij}(x_1) \geq 0, \\ &h_2 = 13 + 2(i-1) + j = 14, \cdots, 19; i = 1, 2, 3; j = 1, 2. \end{aligned} \tag{26}$$

It is found numerically that $\overline{w}_1 = \overline{w}_3 = 1.0000$, $\overline{w}_2 = 0.9052$ such that (24) is satisfied for all i; $\overline{m}_1 = \overline{m}_2 = 1.0000$ such that (25) is satisfied for all j; $\overline{\mu}_{11} = \overline{\mu}_{32} = 1.0000$, $\overline{\mu}_{12} = \overline{\mu}_{31} = 0.1000$, $\overline{\mu}_{21} = \overline{\mu}_{22} = 0.5247$ such that (26) is satisfied for all i and j.

With the third-party Matlab toolbox SOSTOOLS [28], the SOS-based stability conditions in Theorem 2 are employed to check the system stability for $6 \leq a \leq 54$ at the interval of 2 and $10 \leq b \leq 100$ at the interval of 5. With the above settings, choosing $\varepsilon_1(\tilde{\mathbf{x}}) = \varepsilon_{2h_2}(\xi) = \varepsilon_{3h_3}(\xi) = \varepsilon_4(\xi) = 0.0001$; $\mathbf{X}(\tilde{\mathbf{x}})$ is a polynomial of degree 0; $\mathbf{N}_j(x_1)$ as a polynomial of degree 1 in x_1, $\mathbf{R}_{kh_k}(\xi)$ is a polynomial of degree 2 in $(w_1(x_1), w_2(x_1), m_1(x_1), x_1)$, for all h_2, the stability region is shown in Fig. 2 denoted by '$\times$'.

On top of the boundary information of membership grades, we consider the boundary information of premise variables to demonstrate its effect to the stability analysis. Referring to the membership functions of the polynomial fuzzy controller, it is observed that $m_1(x_1) = 0$ for $x_1 \geq 5$ and $m_2(x_1) = 0$ for $x_1 \leq -5$. Based on this information, we consider the following inequalities to bring the boundary information of premise variable x_1 to the stability analysis.

$$\begin{aligned} \gamma_{3h_3}(\xi) &= \varphi_{3h_3}(x_1)\mu_{ij}(x_1) \geq 0, \\ h_3 &= 2(i-1)+j = 1, \cdots, 6, \end{aligned} \tag{27}$$

where $\varphi_{31}(x_1) = \varphi_{33}(x_1) = \varphi_{35}(x_1) = 5 - x_1$ and $\varphi_{32}(x_1) = \varphi_{34}(x_1) = \varphi_{36}(x_1) = x_1 + 5$. Consequently, with the characteristic of the chosen membership functions $m_j(x_1)$, $\gamma_{3h_3}(\xi)$ in (27) demonstrates the property that $\begin{cases} \gamma_{3h_3}(\xi) = 0 \text{ for } x_1 \geq 5 \\ \gamma_{3h_3}(\xi) \leq 0 \text{ for } x_1 \leq 5 \end{cases}$, $h_3 = 1, 3, 5$; $\begin{cases} \gamma_{3h_3}(\xi) \geq 0 \text{ for } x_1 \geq -5 \\ \gamma_{3h_3}(\xi) = 0 \text{ for } x_1 \leq -5 \end{cases}$, h_3 = 2, 4, 6. With the boundary information of premise variables being taken into account, the stability region is shown in Fig. 2 denoted by '$\circ$'. It can be seen that a larger stability region is achieved when both the boundary information of membership grades and premise variables are considered.

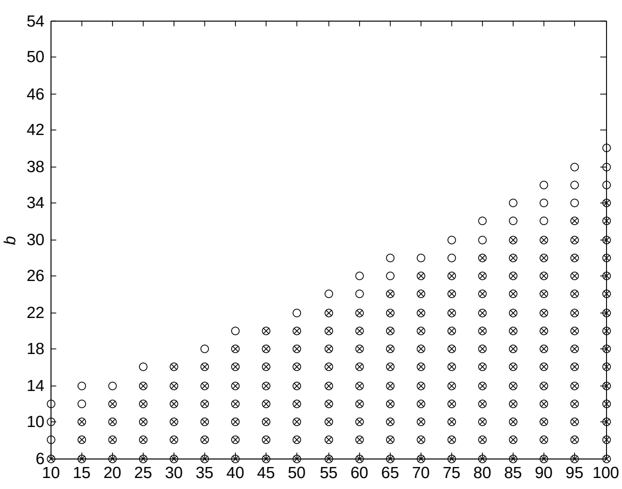

Fig. 2. Stability regions given by Theorem 2 using the boundary information of membership grades only ('$\times$') and plus the boundary information of premise variables ('$\circ$').

To verify the stability analysis result, simulations for the time responses of PFMB control system were conducted. Referring to Fig. 2, considering $a = 100$ and $b = 34$ for the case with only the boundary information of membership grades (stability region indicated by '$\times$'), we obtained $\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} 26.8972 & -0.0792 \\ -0.0792 & 1.1791 \end{bmatrix}$ and the feedback gains as

$$\mathbf{G}_1(x_1) = \begin{bmatrix} -4.4360 & 1.9297 \end{bmatrix}$$

and

$$\mathbf{G}_2(x_1) = \begin{bmatrix} -3.3202 & 2.0437 \end{bmatrix}.$$

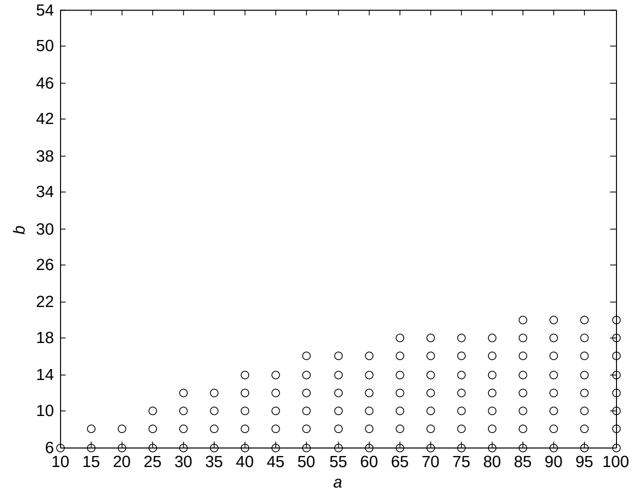

Fig. 3. Stability region given by the SOS-based stability conditions in [30].

It should be noted that although $\mathbf{N}_j(x_1)$ is chosen to be a polynomial of degree 1 in x_1. However, the coefficient corresponding to x_1 is insignificantly small in the the feasible solution and is rounded off to zero.

Referring to Fig. 2, considering $a = 100$ and $b = 40$ for the case with both the boundary information of membership grades and premise variables (stability region indicated by '$\circ$'), we obtained $\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} 81.9556 & -0.8419 \\ -0.8419 & 2.4538 \end{bmatrix}$ and the feedback gains as

$$\mathbf{G}_1(x_1) = \begin{bmatrix} -3.3554 + 0.5250x_1 & 1.6493 + 0.1778x_1 \end{bmatrix}$$

and

$$\mathbf{G}_2(x_1) = \begin{bmatrix} -2.5848 - 0.0423x_1 & 1.6094 - 0.0099x_1 \end{bmatrix}.$$

The phase plots of x_1 and x_2 for both cases subject to various initial conditions are shown in Fig. 4 and Fig. 5, respectively. It can be seen that the PFMB control system for both cases are stable that the polynomial fuzzy controller is able to drive the system states to the origin.

For comparison purposes, the basic SOS-based stability conditions in Theorem 1 are employed to check the system stability. However, no feasible region is found. The SOS-based stability conditions in [30] were employed, which the number of SOS-based stability conditions depends on the number of samples of the premise variables. The number of SOS-based stability conditions in Theorem 2 in this example is 21. By choosing the sampling interval as 1.5385 for x_1, it makes the total number of the SOS-based stability conditions in [30] the same in this example. However, no feasible region can be obtained. When we increase the number of the SOS-based stability conditions in [30] to 24 by choosing the sampling interval as 1.25 for x_1, stability region is found and shown in Fig. 3. The stability analysis result in [30] covers that in [29]. It is thus omitted. Referring to the stability regions in Fig. 2 and Fig. 3, it can be seen that the proposed SOS-based stability conditions in Theorem 2 are able to offer a large size of stability region.

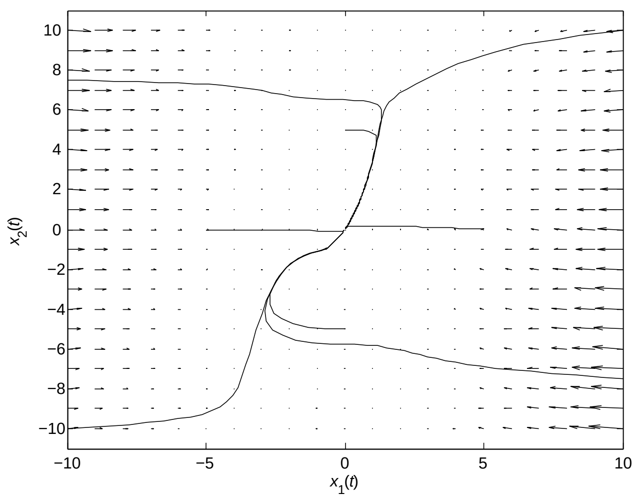

Fig. 4. Phase plot of $x_1(t)$ and $x_2(t)$ for $a = 100$ and $b = 34$ for the case using only the boundary information of membership grades.

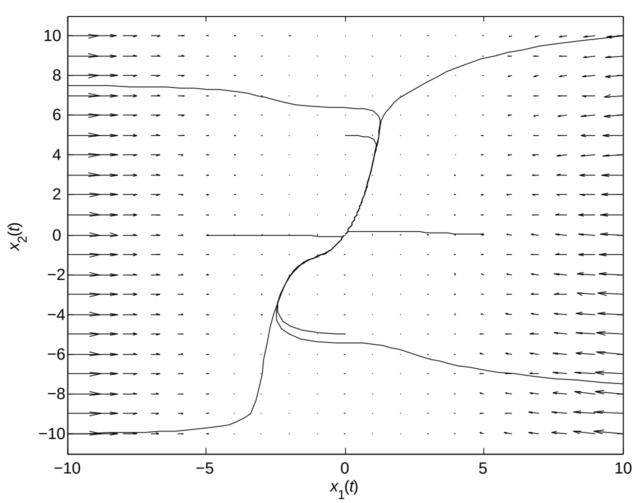

Fig. 5. Phase plot of $x_1(t)$ and $x_2(t)$ for $a = 100$ and $b = 40$ for the case using both the boundary information of membership grades and premise variables.

In order to demonstrate how the membership functions influences the size of stability region, we consider another set of membership functions for the polynomial fuzzy controller, where $m_1(x_1) = \mu_{N_1^1}(x_1) = \begin{cases} \frac{x_1+5}{5} & \text{for } -5 \leq x_1 \leq 0 \\ \frac{-x_1+5}{5} & \text{for } 0 \leq x_1 \leq 5 \\ 0 & \text{for } |x_1| > 5 \end{cases}$ and $m_2(x_1) = \mu_{N_1^2}(x_1) = 1 - m_1(x_1)$.

The inequalities in (24) to (26) are employed to bring the boundary information of membership grades to the stability analysis with $\overline{w}_1 = \overline{w}_3 = 1.0000$, $\overline{w}_2 = 0.9052$; $\overline{m}_1 = \overline{m}_2 = 1.0000$; $\overline{\mu}_{11} = \overline{\mu}_{31} = 0.2000$, $\overline{\mu}_{12} = \overline{\mu}_{32} = 1.0000$, $\overline{\mu}_{21} = 0.9051$, $\overline{\mu}_{22} = 0.3095$ being found numerically. The inequality (27) is to bring the boundary information of premise variables to the stability analysis with $\varphi_{31}(x_1) = \varphi_{33}(x_1) = \varphi_{35}(x_1) = (x_1 + 5)(5 - x_1)$ and $\varphi_{32}(x_1) = \varphi_{34}(x_1) = \varphi_{36}(x_1) = -(x_1 + 5)(5 - x_1)$. By keeping the rest with the same settings for Theorem 2, the stability region is obtained and is shown in Fig. 6. Comparing the stability regions in Fig. 2 and Fig. 6, it can be seen that the second set of membership functions is able to produce a larger size of stability region. It reveals that the second set of membership functions of the polynomial fuzzy controller is more favourable to the control of the nonlinear plant considered in this example. More importantly, it demonstrates that the membership functions play an essential role in the stability analysis of PFMB control systems.

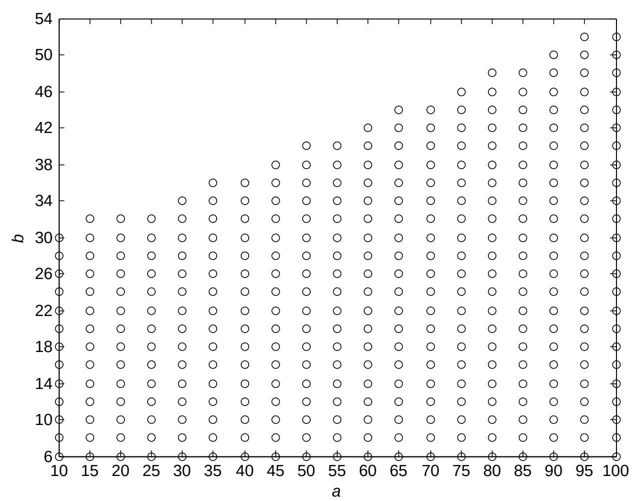

Fig. 6. Stability regions for the PFMB control system with polynomial fuzzy controller using another set of membership functions given by Theorem 2 using both the boundary information of membership grades and the boundary information of premise variables.

V. Conclusion

This paper has investigated the stability of PFMB control systems based on the SOS-based approach using the Lyapunov stability theory. A new SOS-based stability analysis approach has been proposed, which considers three pieces of information, i.e., the property of membership functions, the boundary information of membership grades and premise variables in the stability analysis. SOS-based stability conditions have been obtained to determine the system stability and synthesize the polynomial fuzzy controller. A simulation example has been given to demonstrate the merits of the proposed approach.

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